Graph Theory
Final exam

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11 June 2019

The exam consists of 5 questions worth 10 points each. Choose any 4 questions to answer. If you attempt all 5, the best 4 will count for your final grade. Answer each question carefully and with attention to details, citing any results from the lecture notes that you use. All graphs are simple, finite, and undirected unless otherwise stated. You may write your solutions in Swedish or English. Calculators are not allowed.

Good luck!

1. (Trees)
   (a) Define *forest*. (1)
   (b) Prove that every tree T on n vertices has a vertex x such that $T - \{x\}$ has at least one component with $\leq n/2$ vertices. (2)
   (c) Prove that a minimum weight spanning tree is unique if all the edge weights of the graph are distinct. (3)
   (d) Find all trees that are isomorphic to their complement. (4)

2. (Planarity)
   (a) Define *planar graph*. (1)
   (b) A planar graph is *maximal* if adding any edge makes it nonplanar. Let G be a maximal planar graph with at least 3 vertices. Prove that G does not have a vertex of degree 1. (2)
   (c) Define the *union* of k graphs $G_1, \ldots, G_k$ on the same vertex set to be the graph G with the same vertex set and edges $E(G) = \cup_{i=1}^{k} E(G_k)$. Prove that every planar graph is the union of at most 5 acyclic graphs. (3)
   (d) Let G be a simple planar graph where all vertices have degree at least 5. Prove that G has at least 12 vertices of degree exactly 5. (4)

3. (Colouring)

   (a) Define *chromatic number*. (1)

   (b) Show that for any graph G, there is an ordering of the vertices for which the greedy colouring algorithm uses exactly $\chi(G)$ colours. (3)

   (c) For every $r \geq 2$, construct a graph G with chromatic number at least r which does not contain K_r as a subgraph. (3)

   (d) Construct a graph on countably infinitely many vertices where every vertex has finite degree, but the graph has infinite chromatic number. (3)

4. (Cycles)

   (a) Define *Euler cycle*. (1)

   (b) State and prove a version of Euler's theorem on Euler cycles for directed graphs. (2)

   (c) Prove that if every edge of a graph lies in an odd number of cycles, then the graph is Eulerian. (3)

   (d) Prove that a directed graph in which every vertex has out-degree 1 contains a directed cycle. (4)

5. (Degree problems)

   (a) Define *degree of a vertex*. (1)

   (b) State and prove the handshaking lemma. (2)

   (c) Prove that your friends on average have more friends than you do. That is, let G be a graph on n vertices. Show that

$$\frac{1}{n}\sum_{v\in V(G)} deg(v) \leq \frac{1}{n}\sum_{v\in V(G)}\left(\frac{1}{deg(v)}\sum_{(u,v)\in E(G)} deg(u)\right)$$

   (3)

   (d) Let G be a graph with $n \geq 3$ vertices and $n-1$ distinct vertex degrees. There is some degree i such that two vertices have that degree. What are the possible values for i? (4)